Pergamon

Appl. Math. Lett. Vol. 7, No. 5, pp. 43–47, 1994

0893-9659/94 $7.00 + 0.00

0893-9659(94)00068-9

Solution of a Singular Integral Equation Involving Two Intervals Arising in the Theory of Water Waves

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(Received January 1994; revised and accepted May 1994)

Abstract—Using the well-known inverse of the Cauchy singular integral operator on a single interval along with certain standard results associated with singular integrals, the general solution of a singular integral equation in a double interval arising in the linear theory of water waves, whose kernel involves a logarithmic as well as a Cauchy type singularity, has been obtained. The solutions are found to agree with the known ones.

Keywords—Singular integral equation, Cauchy singular integral operator, Inversion formula, Union of two intervals, Water-wave scattering.

1. INTRODUCTION

The integral equation

$$\int_L f(t)\left[K\ln\left|\frac{x-t}{x+t}\right| + \frac{1}{x+t} + \frac{1}{x-t}\right] dt = g(x), \qquad x \in L, \tag{1.1}$$

where $g(x)$ is a prescribed function, L is the union of two intervals, $(0,a)$ and (b,∞), such that $a < b$, and

$$f(t) = \begin{cases} 0\left(|b-t|^{-1/2}\right), & \text{as } t \to b, \\ 0\left(|a-t|^{-1/2}\right), & \text{as } t \to a, \end{cases} \tag{1.2}$$

arises in the study of scattering of surface water waves (see [1] for similar studies) by a fully immersed vertical plate of finite width. It has been shown recently by Banerjea and Mandal [2] that its solution can be determined by employing an art which is quite interesting but a bit involved. In the present note, we have solved the above integral equation relatively easily by using the standard inversion formula of Cauchy-type singular integral operators of the first kind, the use of which has been demonstrated earlier (see [3,4]) in the context of a certain system of singular integral equations in a single interval. The final solutions agree fully with the ones obtained by Banerjea and Mandal [2]. The present method avoids many details involving complex function theory and the Riemann-Hilbert problem (see [5]) which are customary in handling such singular integral equations. The review article of Estrada and Kanwal [6], as well as a paper by these authors (see [7]), are extremely important in the context of the present integral equation.

We thank the referee whose suggestions have helped in revising the paper.

2. METHOD OF SOLUTION

Let us assume that

$$f(t) = \begin{cases} f_1(t), & \text{for } t \in (0,a), \\ f_2(t), & \text{for } t \in (b,\infty), \end{cases} \tag{2.1}$$

and

$$g(t) = \begin{cases} g_1(t), & \text{for } t \in (0,a), \\ g_2(t), & \text{for } t \in (b,\infty). \end{cases} \tag{2.2}$$

Using the expression (2.1), we write equation (1.1) as

$$\int_0^a f_1(t)\left[K\ln\left|\frac{x-t}{x+t}\right| + \frac{1}{x+t} + \frac{1}{x-t}\right] dt \\ + \int_b^\infty f_2(t)\left[K\ln\left|\frac{x-t}{x+t}\right| + \frac{1}{x+t} + \frac{1}{x-t}\right] dt = g(x). \tag{2.3}$$

After making the substitutions

$$\lambda_1(t) = -K\int_t^a f_1(s)\,ds + f_1(t) \tag{2.4}$$

and

$$\lambda_2(t) = K\int_b^t f_2(s)\,ds + f_2(t), \tag{2.5}$$

the equation (2.3) can be expressed as

$$\int_0^a \frac{\lambda_1(t)}{x^2-t^2}\,dt + \int_b^\infty \frac{\lambda_2(t)}{x^2-t^2}\,dt = \frac{g(x)}{2x}. \tag{2.6}$$

Letting

$$x^2 = \xi, \quad t^2 = \zeta, \quad a^2 = \alpha, \quad b^2 = \beta, \\ \frac{-\lambda_1(t)}{t} = \lambda_1^*(\zeta), \quad \frac{-\lambda_2(t)}{t} = \lambda_2^*(\zeta), \quad \text{and} \quad \frac{g(x)}{x} = g^*(\xi), \tag{2.7}$$

the equation (2.6) can be rewritten as

$$\int_0^\alpha \frac{\lambda_1^*(\zeta)}{\zeta-\xi}\,d\zeta + \int_\beta^\infty \frac{\lambda_2^*(\zeta)}{\zeta-\xi}\,d\zeta = g^*(\xi). \tag{2.8}$$

To solve equation (2.8) we require the following definitions and results.

DEFINITIONS. *We define the operators* $T_1, \tilde{T}_1$, T_2 *and* $\tilde{T}_2$ *as follows:*

(i) $T_1 f = \int_0^\alpha \frac{f(\zeta)}{\zeta-\xi}\,d\zeta$, for $\xi \in (0,\alpha)$.

(ii) $\tilde{T}_1 f = \int_0^\alpha \frac{f(\zeta)}{\zeta-\xi}\,d\zeta$, for $\xi \notin (0,\alpha)$.

(iii) $T_2 f = \int_\beta^\infty \frac{f(\zeta)}{\zeta-\xi}\,d\zeta$, for $\xi \in (\beta,\infty)$.

(iv) $\tilde{T}_2 f = \int_\beta^\infty \frac{f(\zeta)}{\zeta-\xi}\,d\zeta$, for $\xi \notin (\beta,\infty)$.

For simplicity, we use the following notations:

(i) $\Delta_1(\xi) = \sqrt{\frac{\xi}{\alpha-\xi}}$, for $\xi \in (0,\alpha)$, $\hat{\Delta}_1(\xi) = \sqrt{\frac{\xi}{\xi-\alpha}}$, for $\xi \in (\beta,\infty)$.

(ii) $\Delta_2(\xi) = \sqrt{\xi-\beta}$, for $\xi \in (\beta,\infty)$, $\hat{\Delta}_2(\xi) = \sqrt{\beta-\xi}$, for $\xi \in (0,\alpha)$.

(iii) $\Delta_3(\xi) = \sqrt{\xi(\alpha-\xi)}$, for $\xi \in (0,\alpha)$, $\hat{\Delta}_3(\xi) = \sqrt{\xi(\xi-\alpha)}$, for $\xi \in (\beta,\infty)$.

THE INVERSE OPERATORS. *The inverse operators* T_1^{-1} *and* T_2^{-1} *are defined as:*

$$T_1^{-1}h_1 = \frac{C_1}{\Delta_3(\xi)} - \frac{1}{\pi^2 \Delta_1(\xi)} T_1(\Delta_1(\xi)h_1) \tag{2.9}$$

and

$$T_2^{-1}h_2 = \frac{C_2}{\Delta_2(\xi)} - \frac{1}{\pi^2 \Delta_2(\xi)} T_2(\Delta_2(\xi)h_2), \tag{2.10}$$

where C_1 *and* C_2 *are arbitrary constants (see* [3,5]*).*

RESULTS.

1. $T_1^{-1}(T_1 f) = \dfrac{C_1}{\Delta_3(\xi)} + f(\xi).$

2. $T_2^{-1}(T_2 f) = \dfrac{C_2}{\Delta_2(\xi)} + f(\xi).$

3. $T_1^{-1}(\tilde{T}_2 f) = \dfrac{C_1}{\Delta_3(\xi)} - \dfrac{1}{\pi \Delta_1(\xi)} \tilde{T}_2(\hat{\Delta}_1(\xi) f).$

4. $T_2^{-1}(\tilde{T}_1 f) = \dfrac{C_2}{\Delta_2(\xi)} - \dfrac{1}{\pi \Delta_2(\xi)} \tilde{T}_1(\hat{\Delta}_2(\xi) f).$

5. $\tilde{T}_j \left[\dfrac{1}{\Delta_j(\xi)} \tilde{T}_k(\hat{\Delta}_j(\xi) f)\right] = \dfrac{\pi}{\hat{\Delta}_j(\xi)} T_k(\hat{\Delta}_j(\xi) f) - \pi T_k f,$
   where $j \neq k$, $k = 1, 2$ and $j = 1, 2$.

6. $\tilde{T}_j \left[\dfrac{1}{\Delta_j(\xi)} T_j(\Delta_j(\xi) f)\right] = \dfrac{\pi}{\hat{\Delta}_j(\xi)} \tilde{T}_j(\Delta_j(\xi) f), \qquad j = 1, 2.$

7. $T_1 \left(\dfrac{1}{\Delta_1(\xi)}\right) = -\pi.$

8. $\tilde{T}_1 \left(\dfrac{1}{\Delta_1(\xi)}\right) = \pi \left(\dfrac{1}{\hat{\Delta}_1(\xi)} - 1\right).$

9. $\tilde{T}_1 \left(\dfrac{1}{\Delta_3(\xi)}\right) = \dfrac{-\pi}{\hat{\Delta}_3(\xi)}.$

10. $\tilde{T}_2 \left(\dfrac{1}{\Delta_2(\xi)}\right) = \dfrac{\pi}{\hat{\Delta}_2(\xi)}.$

11. $T_1^{-1}(\pi C_2) = \dfrac{C_1}{\Delta_3(\xi)} - \dfrac{C_2}{\Delta_1(\xi)}.$

C_1 and C_2 are arbitrary constants in all the results appearing.

The equation (2.8) can be written as a system of equations as given by

$$T_1 \lambda_1^* + \tilde{T}_2 \lambda_2^* = g_1^*(\xi), \qquad \xi \in (0, \alpha) \tag{2.11}$$

and

$$\tilde{T}_1 \lambda_1^* + T_2 \lambda_2^* = g_2^*(\xi), \qquad \xi \in (\beta, \infty), \tag{2.12}$$

where

$$g_1^*(\xi) = \frac{g_1(x)}{x} \quad \text{and} \quad g_2^*(\xi) = \frac{g_2(x)}{x}. \tag{2.13}$$

Applying the operator T_2^{-1}, as defined by the equation (2.10), on both sides of the equation (2.12) and using the results (2) and (4), we get

$$\lambda_2^* = \frac{1}{\pi\Delta_2(\xi)} \tilde{T}_1 \left(\hat{\Delta}_2(\xi)\lambda_1^*\right) - \frac{1}{\pi^2\Delta_2(\xi)} T_2 \left(\Delta_2(\xi) g_2^*\right) - \frac{C_2}{\Delta_2(\xi)}. \tag{2.14}$$

Using the expression (2.14) in the equation (2.11) along with the results (5), (6) and (10), we get

$$T_1 \left(\hat{\Delta}_2(\xi)\lambda_1^*\right) = \frac{1}{\pi} \tilde{T}_2 \left(\Delta_2(\xi) g_2^*\right) + C_2\pi + g_1^*(\xi)\hat{\Delta}_2(\xi). \tag{2.15}$$

Again, applying the operator T_1^{-1}, on both sides of the equation (2.15), along with the results (1), (3) and (11), we get

$$\lambda_1^* \hat{\Delta}_2(\xi) = \frac{(1+\pi)C_1}{\pi\Delta_3(\xi)} - \frac{C_2}{\Delta_1(\xi)} - \frac{\left[T_1(\Delta_1\hat{\Delta}_2 g_1^*) + \tilde{T}_2(\Delta_2\hat{\Delta}_1 g_2^*)\right]}{\pi^2\Delta_1(\xi)}. \tag{2.16}$$

After little simplification, the equation (2.16) can be expressed as

$$\begin{aligned} \lambda_1^*(\xi) = {} & \frac{-A_1\xi - B_1}{\pi\sqrt{\xi(\alpha-\xi)(\beta-\xi)}} \\ & - \frac{1}{\pi^2\sqrt{\xi}}\sqrt{\frac{\alpha-\xi}{\beta-\xi}} \left[T_1\left(\sqrt{\frac{\beta-\xi}{\alpha-\xi}}\sqrt{\xi}\, g_1^*\right) + \tilde{T}_2\left(\sqrt{\frac{\xi-\beta}{\xi-\alpha}}\sqrt{\xi}\, g_2^*\right)\right] \end{aligned} \tag{2.17}$$

where

$$A_1 = -C_2\pi, \qquad B_1 = C_2\pi\alpha - (1+\pi)C_1, \tag{2.18}$$

and these represent two arbitrary constants. Using the expression on the right of the equation (2.16), in the equation (2.14) and utilizing the results (5), (6), (8) and (9), we arrive at the expression for the unknown function $\lambda_2^*(\xi)$ as given by

$$\begin{aligned} \lambda_2^*(\xi) = {} & \frac{A_1\xi + B_1}{\pi\sqrt{\xi(\xi-\alpha)(\xi-\beta)}} \\ & - \frac{1}{\pi^2\sqrt{\xi}}\sqrt{\frac{\xi-\alpha}{\xi-\beta}} \left[\tilde{T}_1\left(\sqrt{\frac{\beta-\xi}{\alpha-\xi}}\sqrt{\xi}\, g_1^*\right) + T_2\left(\sqrt{\frac{\xi-\beta}{\xi-\alpha}}\sqrt{\xi}\, g_2^*\right)\right]. \end{aligned} \tag{2.19}$$

Thus, the solution of the equation (2.8) is given by the expressions (2.17) and (2.19) for $\xi < \alpha$ and $\xi > \beta$, respectively. Using the transformations as given by the relations (2.7) and (2.13) in the solutions (2.17) and (2.19), we finally obtain the solutions corresponding to the equation (2.6) as given by

$$\lambda_1(x) = \frac{A_1x^2 + B_1}{\pi\sqrt{(a^2-x^2)(b^2-x^2)}} + \frac{2}{\pi^2}\sqrt{\frac{a^2-x^2}{b^2-x^2}}\, P(x), \qquad x < a \tag{2.20}$$

and

$$\lambda_2(x) = \frac{-A_1x^2 - B_1}{\pi\sqrt{(x^2-a^2)(x^2-b^2)}} + \frac{2}{\pi^2}\sqrt{\frac{x^2-a^2}{x^2-b^2}}\, P(x), \qquad x > b, \tag{2.21}$$

where

$$P(x) = \int_0^a \sqrt{\frac{b^2-t^2}{a^2-t^2}}\, \frac{g_1(t).t.dt}{t^2-x^2} + \int_b^\infty \sqrt{\frac{t^2-b^2}{t^2-a^2}}\, \frac{g_2(t).t.dt}{t^2-x^2}. \tag{2.22}$$

Using the expressions (2.4) and (2.5) along with the solutions (2.20) and (2.21), we get the explicit solution of the integral equation (1.1) as given by the formulae

$$f(x) = \begin{cases} f_1(x) = \dfrac{d}{dx}\left[e^{-Kx}\displaystyle\int_a^x e^{Ku}\lambda_1(u)\,du\right], & x < a, \\ f_2(x) = \dfrac{d}{dx}\left[e^{-Kx}\displaystyle\int_b^x e^{Ku}\lambda_2(u)\,du\right], & x > b. \end{cases} \tag{2.23}$$

The results (2.23) agree fully with the ones obtained recently by Banerjea and Mandal [2]. In the case of the homogeneous equation (1.1), for which $g(x) \equiv 0$, the solutions are derivable from the results (2.20) and (2.21), which are in full agreement with the ones obtained by Estrada and Kanwal (see [6,7]).

3. CONCLUSION

The present method of solving the singular integral equation in double interval requires only Cauchy inversion formulae for a single interval and avoids the function theoretic approach and related details.

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